One-Parameter Families of Curves.

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1. It has become an established belief that many of the differential geometric properties of curves and surfaces are more readily studied when the latter are referred to a set of moving axes. In the case of twisted curves the principal directions at a point (*i. e.*, the tangent, principal normal and binormal) afford a good set of axes, and for surfaces it is customary to take the tangent plane at a point for one of the coordinate planes of the axes for this point. In many problems it is convenient to look upon a surface as the locus of a one-parameter family of curves C. In cases of this sort it is often advisable to use for moving axes the principal directions of the curves C. The present paper develops the equations of a surface from this point of view and establishes the fundamental equations of condition to be satisfied by a set of functions determining a surface.

In illustration of the method we consider surfaces: (i) with plane lines of curvature in one system; (ii) with a family of circular generators; (iii) with a family of asymptotic lines of the same constant torsion; (iv) with a family of geodesics of constant torsion.* The general equations lend themselves readily to the consideration of continuous deformations of twisted curves.

If a surface S is generated by a family of curves C, there exist surfaces S' generated by a second family of curves C' such that the tangent, principal normal and binormal at a point of a curve C' are parallel respectively to the binormal, principal normal and tangent at the corresponding point of a curve C. The determination of these surfaces reduces to the integration of a system of ordinary linear differential equations of the first order. The case where S and S' correspond with parallelism of tangent planes is investigated.

General Equations of a Surface.

2. Consider any surface S referred to a family of non-minimal skew curves $v=$const. and any other family of curves $u=$const. We denote by x, y, z the cartesian coordinates of a point M on S, and by α, β, γ; l, m, n; λ, μ, ν the direction-cosines of the tangent, principal normal and binormal at M to the

* These methods have been applied by Dr. R. D. Beetle in an article entitled, "Congruences Associated with a One-Parameter Family of Curves," which will appear in this JOURNAL.

curve $v = \text{const.}$ through M. It is clear that there exist determinate functions p, q, r, t such that

$$\frac{\partial x}{\partial u} = p\,\alpha, \quad \frac{\partial x}{\partial v} = q\,\alpha + r\,l + t\,\lambda, \tag{1}$$

and similar equations in y and z. It is our purpose to determine the conditions which four such functions must satisfy in order that equations of the form (1) define a surface.

In the first place, we observe that the Frenet-Serret formulas* for a curve $v = \text{const.}$ may be written

$$\frac{\partial \alpha}{\partial u} = \frac{p\,l}{\rho}, \quad \frac{\partial l}{\partial u} = -p\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right), \quad \frac{\partial \lambda}{\partial u} = p\,\frac{l}{\tau}, \tag{2}$$

where ρ and τ denote the radii of first curvature and torsion respectively.

The condition of integrability of equations (1) is reducible by means of (2) to

$$\frac{\partial \alpha}{\partial v} = A_1\,\alpha + A_2\,l + A_3\,\lambda, \tag{3}$$

where

$$\left.\begin{aligned} A_1 &= \frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho}, \\ A_2 &= \frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau}, \\ A_3 &= \frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}. \end{aligned}\right\} \tag{4}$$

Since equations analogous to (3) are satisfied by β and γ, and since also

$$\Sigma\,\alpha^2 = 1, \qquad \Sigma\,\alpha\,l = 0, \qquad \Sigma\,\alpha\,\lambda = 0, \tag{5}$$

the function A_1 must be equal to zero.

In like manner the condition of integrability of equation (3) and the first of equations (2) is reducible, by means of (2) and the requirements (5) and

$$\Sigma\,l^2 = 1, \qquad \Sigma\,l\,\lambda = 0, \tag{6}$$

to

$$\frac{\partial l}{\partial v} = -A_2\,\alpha + \left(\frac{\rho}{p}\frac{\partial A_3}{\partial u} - A_2\,\frac{\rho}{\tau}\right)\lambda, \tag{7}$$

and to the additional condition

$$\frac{\partial A_2}{\partial u} - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) + \frac{p}{\tau}\,A_3 = 0. \tag{8}$$

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* E., p. 17. A reference of this kind is to the author's "Differential Geometry," Ginn & Co., Boston, 1909.

Proceeding in a similar manner with equation (7) and the second of equations (2), we find that λ, μ, ν must satisfy equations of the form

$$\frac{\partial \lambda}{\partial v} = - A_3 \alpha - \left(\frac{\rho}{p} \frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau} \right) l, \tag{9}$$

and that the further condition

$$\frac{\partial}{\partial u}\left(\frac{\rho}{p} \frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau} \right) + \frac{\partial}{\partial v}\left(\frac{p}{\tau} \right) + \frac{p}{\rho} A_3 = 0 \tag{10}$$

must be satisfied. It is readily shown that equation (9) and the third of equations (2) are consistent, when the preceding conditions are satisfied.

The equations of condition sought are accordingly $A_1 = 0$, (8) and (10), which in consequence of (4) are

$$\left.\begin{aligned} \frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho} = 0, \\ \frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau} \right) - \frac{\partial}{\partial v}\left(\frac{p}{\rho} \right) + \frac{p}{\tau}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau} \right) = 0, \\ \frac{\partial}{\partial u}\left[\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau} \right) - \frac{\rho}{\tau}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau} \right) \right] + \frac{\partial}{\partial v}\left(\frac{p}{\tau} \right) + \frac{p}{\rho}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau} \right) = 0. \end{aligned}\right\} \tag{11}$$

Retracing the above steps, we see that if the conditions (11) are satisfied by six functions ρ, τ, p, q, r, t, the determination of the surface requires the solution of equations (2), (3), (7) and (9), which is equivalent to the integration of a Riccati equation,* and the quadratures (1).

It is evident at once from equations (1) that the direction-cosines X, Y, Z of the normal to S are of the form

$$X = \sigma (r\lambda - t l), \quad Y = \sigma (r\mu - t m), \quad Z = \sigma (r\nu - t n), \tag{12}$$

where

$$\sigma = (r^2 + t^2)^{-\frac{1}{2}}. \tag{13}$$

Making use of (2), (3), (7) and (9), we obtain from (12) by differentiation

$$\left.\begin{aligned} \frac{\partial X}{\partial u} &= \sigma \left\{ \frac{t p}{\rho} \alpha - \left(p A_3 + \frac{\partial \log \sigma}{\partial u} t \right) l + \left(p A_2 - \frac{p q}{\rho} + r \frac{\partial \log \sigma}{\partial u} \right) \lambda \right\}, \\ \frac{\partial X}{\partial v} &= \sigma \left\{ (t A_2 - r A_3)\, \alpha - \left[\frac{\partial t}{\partial v} + r \left(\frac{p}{\rho} \frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau} \right) + \frac{\partial \log \sigma}{\partial v} t \right] l \right. \\ &\quad \left. + \left[\frac{\partial r}{\partial v} - t \rho \left(\frac{1}{p} \frac{\partial A_3}{\partial u} - \frac{A_2}{\tau} \right) + \frac{\partial \log \sigma}{\partial v} r \right] \lambda \right\}. \end{aligned}\right\} \tag{14}$$

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* Cf. Darboux, "Leçons," Vol. I, p. 56.

Particular Curves Parametric.

3. As immediate consequences of equations (1) we have

THEOREM 1. *The necessary and sufficient condition that the parametric curves form an orthogonal system is that* $q = 0$.

THEOREM 2. *The necessary and sufficient condition that the curves* $v =$ *const. be asymptotic lines is that* $t = 0$.

For in this case the binormals are normal to the surface.

The analytic condition that the parametric curves of a surface form a conjugate system is that*

$$\Sigma \frac{\partial x}{\partial u} \frac{\partial X}{\partial v} = \Sigma \frac{\partial x}{\partial v} \frac{\partial X}{\partial u} = 0.$$

Making use of equations (14) and (4), we have

THEOREM 3. *The necessary and sufficient condition that the parametric curves form a conjugate system is that*

$$t A_2 - r A_3 = \frac{1}{p}\left(t \frac{\partial r}{\partial u} - r \frac{\partial t}{\partial u}\right) + \frac{q t}{\rho} + \frac{1}{\tau}(t^2 + r^2) = 0. \tag{15}$$

If the parametric lines are lines of curvature, equation (15) must hold and also $q = 0$. In this case we have

$$\frac{\partial \omega}{\partial u} - \frac{p}{\tau} = 0, \quad \text{where } \tan \omega = \frac{t}{r}. \tag{16}$$

As thus defined, ω is the angle between the principal normal to a curve $v = \text{const.}$ and the tangent to the curve $u = \text{const.}$ at the same point. Equation (16) expresses the well-known fact that the geodesic torsion is zero.†

When no other condition is put upon the curves $u = \text{const.}$, we can in all generality take $p = 1$. This means that the arcs of the curves $v = \text{const.}$ between two curves $u = \text{const.}$ are equal. With this choice the linear element of S assumes the form

$$d s^2 = d u^2 + 2 q \, d u \, d v + (q^2 + r^2 + t^2) \, d v^2. \tag{17}$$

We observe furthermore that p continues to be $+1$, if the parameters are changed in accordance with the equations

$$u_1 = u + V, \quad v_1 = v, \tag{18}$$

where V denotes any function of v. Under this transformation we have, for the corresponding functions q_1, r_1, t_1, the expressions

$$q_1 = q - V', \quad r_1 = r, \quad t_1 = t, \tag{19}$$

where the prime indicates differentiation.

* E., p. 127.

† Cf. E., p. 138.

As an example, we consider the surfaces whose asymptotic lines in one system have the same constant torsion a. If we put

$$\frac{1}{\tau}=a, \quad t=0, \quad p=1,$$

equations (11) are reducible to

$$\frac{\partial q}{\partial u}-\frac{r}{\rho}=0, \qquad \frac{\partial}{\partial u}\left(\rho\frac{\partial r}{\partial u}+q\right)=0,$$
$$\frac{\partial}{\partial u}\left(\frac{\partial r}{\partial u}+\frac{q}{\rho}\right)-\frac{\partial}{\partial v}\left(\frac{1}{r}\right)-r\,a^2=0.$$

By a proper choice of the parameter u_1, given by (18), and taking this for the new variable u, we may replace the second of the above equations by

$$\rho\frac{\partial r}{\partial u}+q=0.$$

Combining this with the first, we find that q^2+r^2 must be a function of v alone. As this function is the coefficient of dv^2 in the linear element (17), the parameter v can be chosen so that this quantity is equal to unity. Accordingly we can put

$$q=\cos\omega, \quad r=\sin\omega,$$

thus defining the function ω. Now the above equations of condition reduce to

$$\frac{1}{\rho}=-\frac{\partial\omega}{\partial u}, \quad \frac{\partial^2\omega}{\partial u\,\partial v}=a^2\sin\omega,$$

which are the well-known equations for pseudo spherical surfaces.* It can be shown readily that the curves $u=$ const. are asymptotic, and by means of Enneper's theorem we have that they are of constant torsion $-a$.

4. By definition the curves $v=$const. are geodesics if the principal normals to the curves are normal to the surface. Hence, from equations (1) follows

THEOREM 4. *The necessary and sufficient condition that the curves $v=$ const. be geodesics is that $r=0$.*

Suppose that this condition is satisfied. We take the orthogonal trajectories for the curves $u=$const.; then $q=0$. In consequence of the first of equations (11) we may in all generality take $p=1$, and thus the other equations (11) can be written

$$\left.\begin{aligned}\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)&=t\frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)+\frac{2}{\tau}\frac{\partial t}{\partial u},\\ \frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)&=\frac{\partial}{\partial u}\left[\rho\left(\frac{t}{\tau^2}-\frac{\partial^2 t}{\partial u^2}\right)\right]-\frac{1}{\rho}\frac{\partial t}{\partial u}.\end{aligned}\right\} \tag{20}$$

* E., p. 190.

These equations are readily reducible to the form to which the Kowaleski * existence theorem applies. Hence we have

THEOREM 5. *In the problem of finding surfaces whose linear element is of the form $ds^2 = du^2 + t^2 dv^2$, each surface is determined by an arbitrary curve which is taken for the curve $v = 0$; then the intrinsic equations of each curve $v = const.$ are given as power-series in u.* †

When the Curves $v = const.$ Are Plane.

5. The requirement that the curves $v =$ const. be plane (that is, $1/\tau = 0$) affects the general development only in that equation (9) does not arise as in the general case, but because of the conditions

$$\Sigma \alpha \lambda = 0, \quad \Sigma l \lambda = 0.$$

Hence, in order to obtain the equations for plane curves it is necessary and sufficient to put $1/\tau = 0$ in the foregoing equations. In this case equations (11) become

$$\left.\begin{aligned} \frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho} &= 0, \\ \frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho}\right) - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) &= 0, \\ \frac{\partial}{\partial u}\left[\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial t}{\partial u}\right)\right] + \frac{1}{\rho}\frac{\partial t}{\partial u} &= 0. \end{aligned}\right\} \qquad (11')$$

These equations determine t only to within an additive function of v. Consider the cases for t and $t_1 = t + V$. Since p, ρ, τ and r are the same for both, so also are the functions α, β, γ; l, m, n; λ, μ, ν. Hence, from equations (1) we have that the coordinates x, y, z; x_1, y_1, z_1 of the two surfaces are in the relation

$$x_1 - x = \int \lambda V dv, \quad y_1 - y = \int \mu V dv, \quad z_1 - z = \int \nu V dv.$$

We return to the consideration of the general case and write the equation of the plane of a curve $v =$ const. in the form

$$\Sigma x \lambda = V, \qquad (21)$$

where evidently V is a determinate function of v. The characteristic of this plane is defined by (21) and

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* Cf. Picard, "Traité d'Analyse," Vol. II (1893), pp. 318–323.

† The first part of this theorem is in accord with the general theory of the determination of applicable surfaces, since the tangent planes to the desired surface along the curve $v=0$ are tacitly given; they are the rectifying planes of the curve. Cf. Darboux, "Leçons," Vol. II, pp. 263–267.

$$\Sigma x \left[\frac{1}{p} \frac{\partial t}{\partial u} \alpha + \frac{\rho}{p} \frac{\partial}{\partial u} \left(\frac{1}{p} \frac{\partial t}{\partial u} \right) l \right] = - V'. \tag{22}$$

From (21) it follows that this characteristic meets the curve in the points for which $\Sigma \lambda \frac{\partial x}{\partial v} = 0$. Referring to equations (1), we have

THEOREM 6. *The points for which $t = 0$, and only these, lie on the characteristic of the plane of the curves $v = const.$*

In other words:

THEOREM 7. *At a point of intersection of a curve $v = const.$ and of the characteristic of its plane, one of the asymptotic lines is tangent to the curve.*

6. We consider in particular the case where the curves $v =$ const. are plane lines of curvature. We take the orthogonal trajectories for the curves $u =$ const. Hence, in accordance with (16) and a theorem of Joachimsthal,* we have

$$q = 0, \quad r = \cos V \cdot \sigma, \quad t = \sin V \cdot \sigma, \tag{23}$$

where σ is thus defined and V is a function of v alone. For the present we understand that $\cos V \neq 0$, and we introduce a variable v_1 by means of the equation

$$v_1 = \int \cos V \, d v. \tag{24}$$

Equations (11′) may be given the form (dropping the subscript of v_1)

$$\left.\begin{aligned} \frac{\partial \log p}{\partial v} + \frac{\sigma}{\rho} = 0, \\ \frac{\partial}{\partial u}\left(\frac{1}{p} \frac{\partial \sigma}{\partial u} \right) - \frac{\partial}{\partial v}\left(\frac{p}{\rho} \right) = 0, \\ \frac{\partial}{\partial u}\left[\frac{\rho}{p} \frac{\partial}{\partial u}\left(\frac{1}{p} \frac{\partial \sigma}{\partial u} \right) \right] + \frac{1}{\rho} \frac{\partial \sigma}{\partial u} = 0. \end{aligned}\right\} \tag{25}$$

In accordance with the second of these equations we define a function θ by the equations

$$\frac{1}{p} \frac{\partial \sigma}{\partial u} = \frac{\partial \theta}{\partial v}, \quad \frac{p}{\rho} = \frac{\partial \theta}{\partial u}. \tag{26}$$

By means of these equations the last of equations (25) may be reduced to

$$\frac{\partial^2}{\partial u \, \partial v} \log \frac{\partial \theta}{\partial u} + \frac{\partial \theta}{\partial u} \cdot \frac{\partial \theta}{\partial v} = 0, \tag{27}$$

of which the first integral is

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* E., p. 140.

$$\left(\frac{\partial}{\partial v} \log \frac{\partial \theta}{\partial u}\right)^2 + \left(\frac{\partial \theta}{\partial v}\right)^2 = V_1^2, \tag{28}$$

where V_1 denotes an arbitrary function of v alone.

This equation is satisfied by $\dfrac{\partial \theta}{\partial v} = \pm V_1$, but from (26) and the last of (25) it follows that, then, we must have $V_1 = 0$. Hence, σ is a function of v alone, and likewise r and t. Consequently, the curves $u = \text{const.}$ are geodesics, and being lines of curvature at the same time, they are necessarily plane curves. Hence, this case is included in that of $\cos V = 0$, to be considered later.

Excluding this case, we find that the first integral of (28) is

$$\frac{\partial \theta}{\partial v} = V_1 \sin(\theta + V_2), \tag{29}$$

where V_2 is an arbitrary function of v. This equation is reducible to the Riccati form by a suitable change of the dependent variable.

If σ be eliminated from the first equations of (25) and (26), the resulting equation is reducible in consequence of (27) to

$$\frac{\partial}{\partial v}\left(\frac{\partial p}{\partial u} - p \frac{\partial}{\partial u} \log \frac{\partial \theta}{\partial u}\right) = 0.$$

The integral of this equation may be written

$$p = (\phi + V_3) \frac{\partial \theta}{\partial u}, \quad \text{where } \phi = \int \frac{U}{\frac{\partial \theta}{\partial u}} \partial u, \tag{30}$$

U and V_3 being arbitrary functions of u and v respectively. From (25) and (26) it follows that

$$\rho = \phi + V_3, \quad -\sigma = V_1 (\phi + V_3) \cos(\theta + V_2) + \frac{\partial \phi}{\partial v} + V_3', \tag{31}$$

where the prime denotes differentiation.

Equations (2), (3), (7), (9) reduce to

$$\left.\begin{aligned}
&\frac{\partial \alpha}{\partial u} = \frac{\partial \theta}{\partial u} l, && \frac{\partial \alpha}{\partial v} = V_1 \sin(\theta + V_2)(l + \lambda \tan V), \\
&\frac{\partial l}{\partial u} = -\frac{\partial \theta}{\partial u} \alpha, && \frac{\partial l}{\partial v} = -V_1 [\sin(\theta + V_2)\alpha - \tan V \cos(\theta + V_2)\lambda], \\
&\frac{\partial \lambda}{\partial u} = 0, && \frac{\partial \lambda}{\partial v} = -V_1 \tan V [\sin(\theta + V_2)\alpha + \cos(\theta + V_2) l],
\end{aligned}\right\} \tag{32}$$

the function V being an arbitrary function of v alone. Moreover, the equations of the surface are

$$\frac{\partial x}{\partial u} = p\,\alpha, \quad \frac{\partial x}{\partial v} = \sigma (l + \tan V \lambda). \tag{33}$$

From (32) it follows that each solution θ of equation (29) determines the spherical representation of the lines of curvature and that the functions U and V_3 determine a surface of the kind sought with this spherical representation.

In particular, when $U=0$, and only in this case, the curves $v=$ const. are circles, as follows from (31).

Again, in order that the planes of the curves $v=$ const. shall envelop a cylinder whose generators are parallel to the z-axis, it is necessary and sufficient that $\nu=0$. Hence, it follows from the last of equations (32), when λ is replaced by ν, that we may take

$$\gamma=\cos(\theta+V_2), \quad n=-\sin(\theta+V_2).$$

When these values are substituted in the other equations (32), it is found that V_2 is constant, which may in all generality be taken equal to zero. In this case equation (29) can be integrated by a quadrature.

7. We consider now the case $\cos V=0$, which has been excluded. Since the curves $v=$ const. are plane geodesics, S is a surface of Monge.* From the first two of equations (11′) it follows that p and ρ are functions of u alone. This parameter may be chosen so that $p=1$. Since ρ is a function of u alone, the curves $v=$ const. are congruent to one another.

The third of equations (11′) admits the first integral

$$\rho^2\left(\frac{\partial^2 t}{\partial u^2}\right)^2+\left(\frac{\partial t}{\partial u}\right)^2=V^2. \tag{34}$$

This equation is satisfied by $\dfrac{\partial t}{\partial u}=\pm V$, but this is a solution of (11′) only in case $V=0$; that is, when t is a function of v alone. It follows from (17) that S is a developable surface under these conditions.

Excluding this case, the integral of (34) is

$$\frac{\partial t}{\partial u}=V\sin(\theta+V_1),$$

where V_1 is an arbitrary function of v and θ is the function of u defined by

$$\theta=\int\frac{du}{\rho}.$$

Hence, t is given by a quadrature. As surfaces of Monge have been thoroughly discussed by many writers, we will not give any further details.

When the Curves $v=$ const. Are Circles.

8. We choose the curves $u=$ const. so that $p=1$; and we impose the condition $\rho=1/V$, where V is a function of v alone. Hence, the plane curves $v=$ const. are circles. In this case equations (11′) become

* "Application de l'Analyse à la Géométrie," Paris (1849), § 17.

$$\frac{\partial q}{\partial u} - rV = 0, \quad \frac{\partial}{\partial u}\left(\frac{\partial r}{\partial u} + qV\right) - V' = 0, \quad \frac{\partial^3 t}{\partial u^3} + V^2 \frac{\partial t}{\partial u} = 0. \quad (35)$$

Eliminating r from the first two equations, we obtain an equation whose first integral is

$$\frac{\partial^2 q}{\partial u^2} + qV^2 = V(uV' + V_1),$$

where V_1 is an arbitrary function of v. The integral of this equation is

$$q V_2 \sin(Vu + V_3) + \frac{1}{V}(V'u + V_1), \quad (36)$$

where V_2 and V_3 are arbitrary functions of v alone.

From the last of equations (35) we have by integration

$$t = V_4 \sin(uV + V_5) + V_6, \quad (37)$$

V_4, V_5 and V_6 being arbitrary functions of v. Since uV measures the angle between radii of a circle $v = \text{const.}$, it follows from Theorem 6 and equation (37) that, when $V_6 = 0$, the characteristics of the planes of the circles pass through the respective centers.

From the first of (35) we have

$$r = -V_2 \cos(Vu + V_3) + \frac{V'}{V_2}. \quad (38)$$

Hence, the further determination of the surface requires the finding of α, β, γ; $\ldots$; λ, μ, ν, which is reducible to the solution of a Riccati equation.

Since there has been no special choice of v, we may in all generality take $V = v$. Again, in accordance with § 3, we can by a suitable choice of the curves $u = \text{const.}$ reduce V_3 or V_5 to zero. Accordingly we have

THEOREM 8. *The determination of surfaces possessing a family of circles reduces to the integration of a Riccati equation and quadratures; the general solution involves five arbitrary functions of the parameter of the circles.*

The latter part of this theorem is evident also from the point of view of the determination of such a family of circles by the coordinates of the center, the radius and the inclination of the axis of the circle.

Finite and Infinitesimal Deformation of Curves.

9. It is evident that if we take $p = 1$ in the general equations, the arcs of all the curves $v = \text{const.}$ between any two curves $u = \text{const.}$ are equal, and consequently we may look upon the surface as generated by the continuous deformation of one of the curves $v = \text{const.}$

If we write the fundamental equations of § 2 in the form

$$\left.\begin{aligned} &\frac{\partial q}{\partial u}-\frac{r}{\rho}=0, \quad A_2=\frac{\partial r}{\partial u}+\frac{q}{\rho}+\frac{t}{\tau}, \quad A_3=\frac{\partial t}{\partial u}-\frac{r}{\tau}, \\ &\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)=\frac{\partial A_2}{\partial u}+\frac{A_3}{\tau}, \quad \frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)=\frac{\partial}{\partial u}\left(\frac{\rho}{\tau}A_2-\rho\frac{\partial A_3}{\partial u}\right)-\frac{A_3}{\rho}, \end{aligned}\right\} \quad (39)$$

the last two equations give the variation of the intrinsic functions ρ and τ.

General deformations of a twisted curve have little significance, but the interesting cases are those in which all of the curves possess similar properties. For example, in § 3 we treated the case in which a curve of constant torsion is deformed into curves of the same torsion, the direction of deformation of any point of the curve being in its osculating plane at the point; in other words, the osculating planes are tangent to the surface-locus of the curves, and consequently the deformed curves are asymptotic lines.

It is our purpose now to consider the deformations of a curve of constant torsion into curves of constant torsion, the direction of deformation of a point being that of the binormal to the curve at the point. Consequently, the deforms are geodesics on the surface locus of the curves. Surfaces of this kind have been studied by Fibbi* and Bianchi.† We consider the general case for which the torsion varies from curve to curve. Since the curves $v=$ const. are geodesics, we may in all generality take

$$q=r=0, \quad p=1, \quad \frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)=0.$$

The first of equations (39) is satisfied identically; the second may be replaced by

$$\frac{1}{2\rho}=\frac{\partial \theta}{\partial u}, \quad \frac{t}{\tau}=\frac{\partial \theta}{\partial v},$$

θ being a function thus defined; and the last equation may be written

$$\frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)=\frac{\partial}{\partial u}\left[\frac{1}{2\frac{\partial \theta}{\partial u}}\left(\frac{1}{\tau}\frac{\partial \theta}{\partial v}-\tau\frac{\partial^3 \theta}{\partial u^2\,\partial v}\right)\right]-2\tau\frac{\partial \theta}{\partial u}\frac{\partial^2 \theta}{\partial u\,\partial v}.$$

Hence, the problem of the determination of these surfaces reduces to the integration of this equation of the fourth order.

Sannia ‡ has studied at length the infinitesimal deformation of twisted curves. His fundamental equations follow at once from (39). In fact, if q and t are any functions of u, and r is the function of u given by the first of

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* *Annali della Scuola Normale Superiore di Pisa* (1888).

† "Sulla theoria delle trasformazioni delle curve di Bertrand," *Memorie della Società italiana delle Scienze,* Ser. 3, Vol. XVIII.

‡ *Rendiconti del Circolo Matematico di Palermo,* Vol. XXI (1906), pp. 229–256.

(39), we have from the last two of (39) that the intrinsic equations of the new curve are

$$\frac{1}{\rho'} = \frac{1}{\rho} + \left(\frac{\partial A_2}{\partial u} + \frac{A_3}{\tau}\right)\varepsilon, \quad \frac{1}{\tau'} = \frac{1}{\tau} + \left[\frac{\partial}{\partial u}\left(\frac{\rho}{\tau} A_2 - \rho \frac{\partial A_3}{\partial u}\right) - \frac{A_3}{\rho}\right]\varepsilon,$$

where ε denotes an infinitesimal constant.

Surfaces Conjugate to S.

10. If C is a curve whose functions satisfy the Frenet-Serret formulas (2), the functions defined by

$$\alpha' = \lambda, \quad \beta' = \mu, \quad \gamma' = \nu; \qquad l' = -l, \quad m' = -m, \quad n' = -n;$$
$$\lambda' = \alpha, \quad \mu' = \beta, \quad \nu' = \gamma; \qquad \rho' = -\tau, \quad \tau' = -\rho, \quad p' = p, \quad u' = u$$

satisfy similar equations. Hence, these functions define a curve C',* whose tangent, principal normal and binormal are parallel respectively to the binormal, principal normal and tangent of C. We shall show that there exist surfaces S' such that each of its curves $v = \text{const.}$ bears this relation to the corresponding curve on S. Such a surface S' is *conjugate* to S.

If A_2' and A_3' denote functions for S' analogous to A_2 and A_3 for S, the equations analogous to (3), (7) and (9) may be put in the form

$$\frac{\partial \lambda}{\partial v} = -A_2' l + A_3' \alpha, \quad \frac{\partial l}{\partial v} = A_2' \lambda + \left(\tau \frac{\partial A_3'}{\partial u} + A_2' \frac{\tau}{\rho}\right)\alpha,$$
$$\frac{\partial \alpha}{\partial v} = -A_3' \lambda - \left(\tau \frac{\partial A_3'}{\partial u} + A_2' \frac{\tau}{\rho}\right) l.$$

In order that these equations be consistent with (3), (7) and (9), we must have

$$A_2' = \rho\left(\frac{\partial A_3}{\partial u} - \frac{A_2}{\tau}\right), \quad A_3' = -A_3, \quad \tau\left(\frac{\partial A_3'}{\partial u} + \frac{A_2'}{\rho}\right) = -A_2.$$

The last of these equations is a consequence of the first two. Furthermore, when these expressions for A_2' and A_3' are substituted in equations for S' analogous to (8) and (10), we obtain the latter equations in inverse order. Hence, it follows from (4) and (11) that the determination of a surface S' reduces to the solution of the linear system

$$\frac{\partial q'}{\partial u} + \frac{r'}{\tau} = 0, \quad \frac{\partial r'}{\partial u} - \frac{q'}{\tau} - \frac{t'}{\rho} = \rho \frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau}, \quad \frac{\partial t'}{\partial u} + \frac{r'}{\rho} = -A_3. \quad (40)$$

If we compare these equations with (2), we note that if a set of functions q', r', t' satisfy (40), so also do

* Cf. Bianchi, "Lezioni," Vol. I, p. 53.

$$\left.\begin{aligned} q_1' &= q' + V_1 \lambda + V_2 \mu + V_3 \nu, \\ r_1' &= r' - V_1 l - V_2 m - V_3 n, \\ t_1' &= t' + V_1 \alpha + V_2 \beta + V_3 \gamma, \end{aligned}\right\} \tag{41}$$

where V_1, V_2, V_3 are arbitrary functions of v.

From equations (12) it follows that the necessary and sufficient condition that the tangent planes to S and S' at corresponding points be parallel is that

$$\sigma\,(r\lambda - t\,l) = \sigma'\,(r'\alpha + t'\,l), \qquad \sigma\,(r\mu - t\,m) = \sigma'\,(r'\beta + t'\,m),$$
$$\sigma\,(r\nu - t\,n) = \sigma'\,(r'\gamma + t'\,n),$$

from which it follows that

$$r = r' = 0.$$

We may take $q = 0$; then

$$A_2 = \frac{t}{\tau}, \qquad A_3 = \frac{\partial t}{\partial u},$$

and from (40) we have

$$q' = V, \qquad t' = V_1 - t, \qquad -\frac{V}{\tau} + \frac{t - V_1}{\rho} = \rho\left(\frac{\partial^2 t}{\partial u^2} - \frac{t}{\tau^2}\right), \tag{42}$$

where V and V_1 are arbitrary functions of v. The solution of the problem reduces to the determination of three functions ρ, τ, t satisfying equations (11′) and the last of (42). It is readily seen that such solutions exist.

PRINCETON UNIVERSITY, *May* 20, 1914.